# CO-HIGGS BUNDLES ON $\mathbb{P}^1$

STEVEN RAYAN

Abstract. Co-Higgs bundles are Higgs bundles in the sense of Simpson, but with Higgs fields that take values in the tangent bundle instead of the cotangent bundle. Given a vector bundle on $\mathbb{P}^1$, we find necessary and sufficient conditions on its Grothendieck splitting for it to admit a stable Higgs field. We characterize the rank-2, odd-degree moduli space as a universal elliptic curve with a globally-defined equation.



Let $X$ be an algebraic variety with cotangent bundle $T^*$. A Higgs bundle on $X$, in the sense of Simpson [10], is a vector bundle $E \to X$ together with a Higgs field $\phi \in H^0(X; \operatorname{End} E \otimes T^*)$ for which $\phi \wedge \phi = 0 \in H^0(X; \operatorname{End} E \otimes \wedge^2 T^*)$. Higgs bundles have been studied intensely, and appear naturally in areas of mathematics as diverse as string theory and number theory—see [2] for an overview.

An alternative kind of Higgs bundle arises when we replace $T^*$ with $T$ in the definition of the Higgs field. These objects, which we call *co-Higgs bundles*, are only beginning to attract interest. Some preliminary discussions appear in [6, 7]. One motivation for studying co-Higgs bundles comes from generalized geometry, because generalized holomorphic bundles on ordinary complex manifolds are co-Higgs bundles [4].

The purpose of this note is to characterize co-Higgs bundles over curves. In this case, $\phi \wedge \phi = 0$ is automatic. By a "curve" we will always mean a nonsingular, connected, projective curve over $\mathbb{C}$.

We show that stability restricts our study to the projective line. We then characterize the vector bundles on $\mathbb{P}^1$ admitting semistable Higgs fields in terms of their splitting types, and use this to study the odd-degree component of the rank-2 moduli space explicitly. The main result is a global description of this smooth moduli space as the variety of solutions of an algebraic equation. This equation is a universal one for the fibres of the associated Hitchin map, whose generic fibre in this case is a nonsingular elliptic curve. In the even case, we characterize a section of the fibration by the splitting type of $E$.

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## 1. MORPHISMS, STABILITY, AND $S$-EQUIVALENCE

The following notions carry over from Higgs bundles without modification. A morphism taking $(E, \phi)$ to $(E', \phi')$ is a commutative diagram

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$$\begin{array}{ccc} E & \xrightarrow{\psi} & E' \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi'} \\ E\otimes K^* & \xrightarrow{\psi\otimes 1} & E'\otimes K^* \end{array}$$

in which $\psi : E \to E'$ is a morphism of vector bundles. (The tangent bundle is denoted by $K^*$ here, as it is the anticanonical line bundle of $X$.) The pairs $(E, \phi)$ and $(E', \phi')$ are isomorphic if $\psi$ is an isomorphism of bundles. In particular, $(E, \phi)$ and $(E, \phi')$ are isomorphic if and only if there exists an automorphism of $\psi$ of $E$ such that $\psi\phi\psi^{-1} = \phi'$.

The appropriate stability condition for the moduli space is Hitchin's slope-stability condition for Higgs bundles on curves [5].

**Definition 1.1.** A co-Higgs bundle $(E, \phi)$ over $X$ is *(semi)stable* if

$$\frac{\deg U}{\operatorname{rk} U} < \frac{\deg E}{\operatorname{rk} E} \tag{1.1}$$

(respectively, $\le$) for each proper subbundle $U \subset E$ that is invariant under $\phi$ (meaning $\phi U \subseteq U \otimes K^*$). The rational number $\mu(U) := \deg U / \operatorname{rk} U$ is called the *slope* of $U$.

Clearly, if $E$ is stable as a vector bundle—meaning that all of its subbundles satisfy (1.1)—then for any Higgs field $\phi \in H^0(X; \mathrm{End}\ E \otimes K^*)$, the pair $(E, \phi)$ is also stable.

**Remark.** An important property of stable co-Higgs bundles is that they are *simple*: every morphism of a stable $(E, \phi)$ is a multiple of the identity. In other words, every endomorphism of $E$ that commutes with $\phi$ acts on $E$ by rescaling only. The proof of this fact can be adapted immediately from the analogous result for stable vector bundles; see for instance [8].

If $(E, \phi)$ is semistable but not stable, $E$ has a proper subbundle $U$ for which $(U, \phi)$ is stable. It follows that $(E/U, \phi)$ is semistable. This process, which terminates eventually, gives us a *Jordan-Hölder filtration* of $E$:

$$0 = E_0 \subset \cdots \subset E_m = E,$$

for some $m$, where $(E_j, \phi)$ is semistable for $1 \le i \le m - 1$, and where $(E_j/E_{j-1}, \phi)$ is stable and $\mu(E_j/E_{j-1}) = \mu(E)$ for $1 \le j \le m$. While this filtration is not unique, the isomorphism class of the following object is:

$$\mathrm{gr}(E, \phi) \ := \ \bigoplus_{j=1}^{m} (E_j/E_{j-1}, \phi).$$

This object is called the *associated graded object* of $(E, \phi)$. Then, two semistable pairs $(E, \phi)$ and $(E, \phi')$ are said to be *S-equivalent* whenever $\mathrm{gr}(E, \phi) \cong \mathrm{gr}(E', \phi')$. If a pair is strictly stable, then the underlying bundle has the trivial Jordan-Hölder filtration consisting of itself and the zero bundle, and so the isomorphism class of the graded object is nothing more than the isomorphism class of the original pair.

## 2. HIGHER GENUS

Stable co-Higgs bundles with sufficiently interesting Higgs fields only occur on the projective line. To see this, suppose that $X$ has genus $g > 1$ and that $(E, \phi)$ is a stable co-Higgs bundle on $X$. The canonical line bundle $K$ has $g$ sections: choose one, say, $s$. Taking the product $s\phi$ contracts $K$ with $K^*$; that is, $s\phi$ is an endomorphism of $E$. But $s\phi$ and $\phi$ commute, and so $s\phi$ must be a multiple of the identity, by the "simple" property of stability. Because $\deg K = 2g - 2 > 1$, $s$ vanishes somewhere, and so $\phi$ must vanish everywhere. In other words, a stable co-Higgs bundle on $X$ with $g > 1$ is nothing more than a stable vector bundle.

When $g = 1$, co-Higgs bundles are Higgs bundles.

This leaves only the projective line. We will see that stable co-Higgs bundles with non-scalar Higgs fields are plentiful here. This is in contrast to Higgs bundles, which are never stable on $\mathbb{P}^1$. Co-Higgs bundles, therefore, are an extension of the theory of Higgs bundles to genus 0.

## 3. NITSURE'S MODULI SPACE

For the moduli space, we rely on [9], in which Nitsure constructs a quasiprojective variety that is a coarse moduli space for $S$-equivalence classes of semistable "$L$-pairs" of rank $r$ on a curve $X$. Here, $L$ is a sufficiently-ample line bundle and "$L$-pair" means a pair $(E, \phi)$ in which $E$ is a rank-$r$ vector bundle and $\phi \in H^0(X; \operatorname{End} E \otimes L)$. The construction uses geometric invariant theory, and the stability condition is the one defined previously. For $X = \mathbb{P}^1$ and $L = \mathcal{O}(2)$, we have the moduli space of semistable co-Higgs bundles on the projective line. We use $\mathcal{M}(r)$ to signify this space; $\mathcal{M}(r, d)$, the component in $\mathcal{M}(r)$ consisting of degree-$d$ co-Higgs bundles. When $r$ and $d$ are coprime, $\mathcal{M}(r, d)$ is smooth.

For $r = 2$, we need only describe the components $\mathcal{M}(r, -1)$ and $\mathcal{M}(r, 0)$, as we can recover co-Higgs bundles of other degrees by tensoring the elements of these two spaces by $\mathcal{O}(\pm 1)^{\otimes n}$ for appropriate $n$. In [9] Nitsure calculates the dimension of $\mathcal{M}(r)$ to be $2r^2 + 1$, and so $\mathcal{M}(2)$ is 9-dimensional. (He proves that the dimension is independent of the degree component.) For a simplification, we consider only trace-free Higgs fields. The map

$$\mathcal{M}(2) \quad \rightarrow \quad H^0(\mathbb{P}^1; \mathcal{O}(2)) \times \mathcal{M}_0(2)$$

defined by

$$(E, \phi) \quad \mapsto \quad \left( \operatorname{Tr} \phi, \left( E, \phi - \frac{1}{2} \operatorname{Tr} \phi \right) \right),$$

where $\mathcal{M}_0(2)$ denotes the 6-dimensional trace-free part of the moduli space, is an isomorphism. As $\operatorname{Tr} \phi$ is a Higgs field for a line bundle, the factorization can be thought of as $\mathcal{M}(2) = \mathcal{M}(1) \times \mathcal{M}_0(2)$, where the first factor is the space of co-Higgs line bundles. The piece of the moduli space that we do not already understand is $\mathcal{M}_0(2)$, and so there is no generality lost in restricting attention to it.

## 4. HITCHIN MORPHISM AND SPECTRAL CURVES

Consider the *Hitchin map* $h : \mathcal{M}(r) \to \bigoplus_{k=1}^r H^0(\mathbb{P}^1; \mathcal{O}(2k))$ given by $(E, \phi) \mapsto \operatorname{char} \phi$, where $\operatorname{char} \phi$ is the characteristic polynomial of $\phi$. Since $\operatorname{char} \phi$ is invariant under conjugation, this map is well-defined on equivalence classes. Nitsure proves in [9] that $h$ is proper. In particular, preimages of points are compact. Therefore, the fibres of $h$ are compact.

Let $\rho \in \bigoplus_{k=1}^r H^0(\mathbb{P}^1; \mathcal{O}(2k))$ be a generic section. It follows from more general arguments in [1] and [3] that the fibre $h^{-1}(\rho)$ is isomorphic to the Jacobian of a *spectral curve* embedded as a smooth subvariety $X_\rho$ of the total space of $\mathcal{O}(2)$. The correspondence works like this:

(a) if $\pi$ is the projection to $\mathbb{P}^1$ of the total space of $\mathcal{O}(2)$, then the restriction $\pi_\rho : X_\rho \to \mathbb{P}^1$ is an $r : 1$ covering map;
(b) the equation of $X_\rho$ is $\rho(\pi(\eta)) = 0$, where $\eta$ is the tautological section of the pullback of $\mathcal{O}(2)$ to its own total space;
(c) the direct image of a line bundle $L$ on a generic $X_\rho$ is a rank-$r$ vector bundle $(\pi_\rho)_* L = E$ on $\mathbb{P}^1$;
(d) the pushforward of the multiplication map $L \to \eta L$ is a Higgs field $\Phi$ for $E$, with characteristic polynomial $\rho$.

The spectral curve ramifies at finitely-many points, which are the $z \in \mathbb{P}^1$ for which $\phi_z$ has repeated eigenvalues. The generic characteristic polynomial $\rho$ is irreducible, and so its $X_\rho$ is connected.

In the case of rank $r = 2$ and $\phi$ trace-free, the characteristic polynomial is a monic polynomial of degree 2 in $\eta$ with no linear term, and with a section of $\mathcal{O}(4)$ for the coefficient of $\eta^0$. This section vanishes at 4 generically distinct points in $\mathbb{P}^1$, which are the ramification points of the double cover $X_\rho \to \mathbb{P}^1$. By the Riemann-Hurwitz formula, $X_\rho$ is an elliptic curve, whose Jacobian is another elliptic curve. Therefore, the map $h$ on the rank-2 trace-free moduli space is a fibration of generically nonsingular elliptic curves over the 5-dimensional affine space of determinants.

Because the generic $\rho$ is irreducible, a co-Higgs bundle $(E, \phi)$ coming from a line bundle on $X_\rho$ has no $\phi$-invariant subbundles, and so is stable. Stability limits the underlying vector bundles that can be obtained from spectral line bundles. In the next section, we address this.

## 5. STABLE GROTHENDIECK NUMBERS

According to the classical Birkhoff-Grothendieck theorem, if $E$ is a rank-$r$ vector bundle on $\mathbb{P}^1$, then $E \cong \mathcal{O}(m_1) \oplus \mathcal{O}(m_2) \oplus \cdots \mathcal{O}(m_r)$ for integers $m_1, m_2, \ldots, m_r$ that are unique up to permutation. For $E = \mathcal{O}(m_1) \oplus \cdots \oplus \mathcal{O}(m_r)$ on $\mathbb{P}^1$, we find necessary and sufficient conditions on the Grothendieck numbers $m_i$ for the existence of stable Higgs fields.

**Theorem 5.1.** *A rank-$r$ vector bundle*

$$E = \mathcal{O}(m_1) \oplus \mathcal{O}(m_2) \oplus \cdots \oplus \mathcal{O}(m_r)$$

*over $\mathbb{P}^1$, where $m_1 \geq m_2 \geq \cdots \geq m_r$, admits a semistable $\phi \in H^0(\mathbb{P}^1; \operatorname{End} E \otimes \mathcal{O}(2))$ if and only if $m_i \leq m_{i+1} + 2$ for all $1 \leq i \leq r-1$. The generic $\phi$ leaves invariant no subbundle of $E$ whatsoever; therefore, the generic $\phi$ is stable trivially.*

*Proof.* Since every co-Higgs line bundle is stable, we consider only $r > 1$. We begin with the *only if* direction, for which we proceed by induction on successive extensions of balanced bundles by each other. (A rank-$r$ *balanced vector bundle* over $\mathbb{P}^1$ splits into $r$ copies of a single line bundle.) To arrive at these bundles, we filter the decomposition of $E$ by its repeated Grothendieck numbers. That is, if the first $d_1$ ordered Grothendieck numbers are $m_1 = \cdots = m_{d_1} = a_1$, then we write $E_1$ for the balanced vector bundle $\bigoplus^{d_1} \mathcal{O}(a_1)$. If the next $d_2$ numbers are all equal to the same number, say $a_2$, then we set $E_2 := \bigoplus^{d_2} \mathcal{O}(a_2)$; and so on. Then, $E = \bigoplus_{i=1}^k E_i = \bigoplus_{i=1}^k \left( \bigoplus^{d_i} \mathcal{O}(a_i) \right)$, where $d_1 + \cdots + d_k = r$ and $a_1 > \cdots > a_k$.

Begin with the (inexact) sequence

$$E_1 \xrightarrow{\phi} E \otimes \mathcal{O}(2) \xrightarrow{p} (E_2 \oplus \cdots \oplus E_k) \otimes \mathcal{O}(2).$$

The composition of $\phi$ with the quotient map $p$ is a section of $E_1^* \otimes E/E_1 \otimes \mathcal{O}(2)$, and so has components in $\mathcal{O}(-a_1 + a_j + 2)$, for each of $j = 2, 3, \ldots, k$. If $a_1 > a_2 + 2$, then $a_1 > a_j + 2$ for $j = 2, 3, \ldots, k$ and

$$H^0(\mathbb{P}^1; \mathcal{O}(-a_1 + a_2 + 2)) \ = \ \cdots \ = \ H^0(\mathbb{P}^1; \mathcal{O}(-a_1 + a_k + 2)) \ = \ 0.$$

Therefore, $p \circ \phi$ is the zero map. It follows that $E_1$ is $\phi$-invariant. It is destabilizing, however, because $d_1 + \cdots + d_k = r$ and $a_1 > a_2 > \cdots > a_k$ imply

$$\frac{\deg E_1}{\operatorname{rk} E_1} \ = \ \frac{d_1 a_1}{d_1} \ = \ a_1 = \frac{a_1(d_1 + \cdots + d_k)}{r} > \frac{d_1 a_1 + d_2 a_2 + \cdots d_k a_k}{r} \ = \ \frac{\deg E}{\operatorname{rk} E}.$$

In light of the contradiction, we must have $a_1 \leq a_2 + 2$, and so

$$m_1 = \cdots = m_{d_1} \ \leq \ m_{d_1+1} + 2 = \cdots = m_{d_1+d_2} + 2.$$

We assume now that

$$\begin{aligned} a_2 &\leq a_3 + 2 \\ &\vdots \\ a_{j-1} &\leq a_j + 2. \end{aligned}$$

We examine the sequence

$$E_1 \oplus E_2 \oplus \cdots \oplus E_j \xrightarrow{\phi} E \otimes \mathcal{O}(2) \xrightarrow{p} (E_{j+1} \oplus \cdots \oplus E_k) \otimes \mathcal{O}(2),$$

where we abuse notation and re-use $p$ for the quotient of $E$ by $E_1 \oplus \ldots \oplus E_j$. We assume that $a_j > a_{j+1} + 2$. Because of the induction hypothesis, we have that $a_i > a_j > a_u + 2$ for each $i \leq j$ and each $u > j$. Therefore, $-a_i + a_u + 2 < 0$, and the images of the balanced bundles $E_i$, $i \leq j$, are zero under the composition of $\phi$ and $p$. Hence, $E_1 \oplus \cdots \oplus E_j$ is $\phi$-invariant and its slope exceeds that of $E$. The induction is complete.

**Remark.** The above argument does not rely on $X = \mathbb{P}^1$: $X$ could be projective space $\mathbb{P}^n$ of any dimension, so long as we are considering fully decomposable bundles. In that case,

the result would say that stable Higgs fields exist only if $m_i \leq m_{i+1}+s$, where $s$ is the largest integer such that $T(s)$ has sections.

Conversely, suppose that $m_i \leq m_{i+1}+2$ for each $i = 1, \ldots, r-1$. Our strategy is to find a particular Higgs field $\phi$ under which no subbundle of $E$ is invariant, meaning that $(E, \phi)$ is trivially stable. Because of the decomposition of $E$ into a sum of line bundles $\mathcal{O}(m_i)$, we can realize the Higgs field is a $r \times r$ matrix whose $(i,j)$-th entry takes values in the line bundle $\mathcal{O}(-m_j+m_i+2)$. Let us consider the $(r-1)\times(r-1)$ matrix that remains when we ignore the first row and the $r$-th column. The diagonal elements of this matrix are sections of $\mathcal{O}(-m_i+m_{i-1}+2) \cong \mathcal{O}(p_i)$ for $i = 2, \ldots, r$, where each $p_i$ is one of 0, 1, or 2. Into each of these positions, we enter a '1', which in the case of the trivial line bundle ($p_i = 0$) is simply the number 1. In the case of $p_i = 1$, '1' represents the section of $\mathcal{O}(1)$ that is 1 on $\mathbb{P}^1 - \infty$ and is $1/z$ on $\mathbb{P}^1 - 0$, where $z$ is the affine parameter on $\mathbb{P}^1 - \infty$. For $\mathcal{O}(2)$, '1' refers to the section that is 1 on $\mathbb{P}^1 - \infty$ and $1/z^2$ on $\mathbb{P}^1 - 0$. In each case, 1 is well-defined. For all other entries of the $(r-1)\times(r-1)$ sub-matrix, we insert the zero section of the corresponding line bundle. For the first row and $r$-th column, we insert zeros everywhere save for the $(1, r)$-th entry $q \in H^0(\mathbb{P}^1; \mathcal{O}(-m_r + m_1 + 2))$. If $q$ is not identically zero, then it is a polynomial in $z$ of degree two or more. The degree of $q$ is precisely two when the bundle is balanced (although zero can be taken for the leading coefficient), and the maximal degree increases by one or two for each respective jump between the Grothendieck numbers. In any case, we can always choose $q(z) = z$. The characteristic polynomial of $\phi$ is therefore $-z + y^r$, which is irreducible over $\mathbb{C}[y][z]$.

$$\phi(z) \quad = \quad \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & z \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

Because the characteristic polynomial does not split, $\phi$ has no proper eigen-subbundles in $E$; that is, $E$ has no $\phi$-invariant subbundles. As irreducibility is an open condition, the genericity follows immediately: there is a Zariski open subset of $H^0(\mathbb{P}^1; \mathrm{End}\ E \otimes \mathcal{O}(2))$ whose elements leave invariant no subbundles whatsoever. $\square$

For the case of rank $r = 2$, Theorem 5.1 tells us if $E$ has degree 0, then $E \to \mathbb{P}^1$ admits (semi)stable Higgs fields if and only if $E \cong \mathcal{O} \oplus \mathcal{O}$ or $E \cong \mathcal{O}(1) \oplus \mathcal{O}(-1)$. On the other hand, if $E$ has degree -1, there is only one choice: $\mathcal{O} \oplus \mathcal{O}(-1)$.

## 6. ODD DEGREE

We examine $\mathcal{M}_0(2, -1)$, where the underlying bundle of every co-Higgs bundle is isomorphic to $E = \mathcal{O} \oplus \mathcal{O}(-1)$. Since $E$ has non-integer slope, every semistable Higgs field for $E$ is stable. Every Higgs field for $E$ is of the form

$$\phi \quad = \quad \begin{pmatrix} a & b \\ c & -a \end{pmatrix},$$

where $a$, $b$, and $c$ are sections of $\mathcal{O}(2)$, $\mathcal{O}(3)$, and $\mathcal{O}(1)$, respectively. The stability of $\phi$ means that $c$ is not identically zero: because $\mu(E) = -1/2$, $\phi$ cannot leave the trivial sub-line bundle $\mathcal{O}$ invariant. Accordingly, $c$ has a unique zero $z_0 \in \mathbb{P}^1$.

It is possible to provide a global description of the odd-degree moduli space as a universal elliptic curve. Let $M$ stand for the two-dimensional total space of $\mathcal{O}(2)$, and let $\pi : M \to \mathbb{P}^1$ be the natural map. We claim that we can assign uniquely to each stable $\phi$ a point in the 6-dimensional space $\mathcal{S}$ defined by

$$\left\{ (y,\rho) \in M \times H^0(\mathbb{P}^1;\mathcal{O}(4)) \; : \; \eta^2(y) = \rho(\pi(y)) \right\}.$$

That $\mathcal{S}$ is a smooth subvariety of the 7-dimensional space $M \times H^0(\mathbb{P}^1;\mathcal{O}(4))$ can be seen as follows. Over the affine patch $U_0$ of $\mathbb{P}^1$ where the coordinate $z$ is not $\infty$, we have

$$\mathcal{S} = \left\{ (z, y, a_0, \ldots, a_4) \; : \; y^2 = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 \right\},$$

with $y$ as the vertical coordinate on $M$. If $\tilde{z} = 1/z$ and $\tilde{y} = y/z^2$, then $(\tilde{z}, \tilde{y})$ give coordinates on $M$ over $U_1 = \mathbb{P}^1 - 0$. There, $\mathcal{S}$ is given by $\tilde{y}^2 = a_4 + a_3\tilde{z} + \cdots + a_0\tilde{z}^4$. Since $\partial f/\partial a_0 \neq 0$ on $M|_{U_0} \times \mathbb{C}^5$ and $\partial \tilde{f}/\partial a_4 \neq 0$ on $T|_{U_1} \times \mathbb{C}^5$, where $f(z, y, a_0, \ldots, a_4) = y^2 - a_0 - a_1 z - ... - a_4 z^4$ and $\tilde{f}(\tilde{z}, \tilde{y}, a_0, \ldots, a_4) = \tilde{y}^2 - a_4 - a_3\tilde{z} - ... - a_0\tilde{z}^4$, the variety $\mathcal{S}$ is in fact smooth as a subvariety.

We will define an isomorphism of $\mathcal{M}_0(2,-1)$ onto $\mathcal{S}$ by sending $\phi$ to $(a(z_0), -\det\phi)$, with $z_0$ and $a$ as above. Since $a$ is a section of $\mathcal{O}(2)$, $a(z_0)$ is a point on $M$. The point is determined uniquely by the conjugacy class of $\phi$, for if

$$\psi = \begin{pmatrix} 1 & d \\ 0 & e \end{pmatrix}$$

is an automorphism of $E$, in which case $d$ is a section of $\mathcal{O}(1)$ and $e \neq 0$ is a number, then the Higgs field transforms as

$$\phi' = \psi\phi\psi^{-1} = \begin{pmatrix} a + dc & -e^{-1}(2ad - b + cd^2) \\ ec & -a - dc \end{pmatrix}.$$

Because $(a + dc)(z_0) = a(z_0)$, the image of $\phi$ in the variety $\mathcal{S}$ remains unchanged by $\phi \to \phi'$. Since $c$ vanishes at $z_0$, we have $(a(z_0))^2 = -\det\phi(z_0)$, and therefore $(a(z_0), -\det\phi)$ is a point on $\mathcal{S}$.

For the other direction, we begin with a point $(y_0, \rho_0) \in \mathcal{S} \subset M \times \mathbb{C}^5$. Choose an affine coordinate $z$ on $\mathbb{P}^1$ such that $\pi(y_0) = 0$ in this coordinate. By the definition of $\mathcal{S}$, $y_0^2 = \rho_0(0)$, and so we may write $\rho_0(z) = y_0^2 + zb(z)$ for $b(z)$ a cubic polynomial. This makes

$$\phi(z) = \begin{pmatrix} y_0 & b(z) \\ z & -y_0 \end{pmatrix}$$

a representative Higgs field.

**Remark.** For a fixed generic $\rho$, the points $(y, \rho)$ on $\mathcal{S}$ are the points of the spectral curve $X_\rho$. According to Riemann-Roch, to get $(\pi_\rho)_* L = \mathcal{O} \oplus \mathcal{O}(-1) = E$ on $\mathbb{P}^1$ we need a line bundle $L$ of degree 1 on $X_\rho$. Also by Riemann-Roch, every such line bundle has a 1-dimensional space of sections, and so there is a single point in $X_\rho$ at which all of the sections vanish. Now, pulling back $(\pi_\rho)_* L = E$ to $X_\rho$ gives an evaluation map $\pi_\rho^* E \to L$, whose kernel is a line bundle on $X_\rho$. This is the bundle of eigenspaces in $E$ with respect to

$\phi = (\pi_\rho)_*(L \to \eta L)$. The maximal destabilizing subbundle $\mathcal{O}$ of $E$ is preserved by $\phi$ when the evaluation map restricted to $\mathcal{O}$ is zero. But this is the vanishing of the unique section of $L$. Consequently, the defining point of $L$ is the eigenvalue of $\phi$ at the point where $\mathcal{O}$ is an eigenspace. This point is $z_0$, and so $L$ is given by the point $a(z_0)$ on $X_\rho$.

## 7. EVEN DEGREE

The co-Higgs moduli space with degree-0 underlying bundle does not yield such an explicit description; however, we can still say something about the fibres of the Hitchin map.

Recall that Theorem 5.1 allows for two choices of underlying bundle: $E_{-1}^{\,1} := \mathcal{O}(1)\oplus\mathcal{O}(-1)$ or the trivial rank-2 bundle $E_0 := \mathcal{O} \oplus \mathcal{O}$, the latter of which is the generic splitting type. If a pair $(E_{-1}^{\,1}, \phi)$ is *not* unstable, then it is strictly stable: any subbundle of nonnegative degree must be isomorphic to $\mathcal{O}(1)$, and so the pair can have no destabilizing subbundle of degree 0. On the other hand, $E_0$ admits semistable but not stable fields $\phi$: these are the upper-triangular Higgs fields, in which the three matrix coefficients in the polynomial $\phi(z) = A_0 + A_1 z + A_2 z^2$ admit a common eigenvector. The $S$-equivalence class of such a $\phi$ is represented by its associated graded object

$$\mathrm{gr}(\phi) \;=\; \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix},$$

for some $a \in H^0(\mathbb{P}^1; \mathcal{O}(2))$. This form is fixed by the determinant $\rho = -a^2$ and so any fibre of the Hitchin map has at most one semistable but not stable Higgs field. In fact, the generic fibre has none, because $\rho = -a^2$ is a disconnected spectral curve. One example of a nongeneric fibre is the nilpotent cone over $\rho = 0$: in addition to stable Higgs fields it also contains the zero Higgs field for $E_0$, which is semistable but not stable.

To study Higgs fields for $E_{-1}^{\,1}$, we define a section of the Hitchin map $h : \mathcal{M}_0(2,0) \to H^0(\mathbb{P}^1; \mathcal{O}(4))$ in the following way: to each $\rho \in H^0(\mathbb{P}^1; \mathcal{O}(4))$, we assign the Higgs field for $E_{-1}^{\,1}$

$$Q(\rho) \;=\; \begin{pmatrix} 0 & -\rho \\ 1 & 0 \end{pmatrix},$$

with the symbol 0 denoting the zero section of $\mathcal{O}(2)$, and where 1 is unity. This section is the genus-0 analogue of Hitchin's model of Teichmüller space [5], but with our $\rho$ replacing the quadratic differential in his model.

**Proposition 7.1.** *The section $Q$ is the locus in $\mathcal{M}_0(2,0)$ of co-Higgs bundles with underlying bundle isomorphic to $E_{-1}^{\,1} = \mathcal{O}(1) \oplus \mathcal{O}(-1)$.*

*Proof.* If

$$\phi \;=\; \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

is a Higgs field for $E_{-1}^{\,1}$, then $a$ is a section of $\mathcal{O}(2)$ and $b$ is a section of $\mathcal{O}(4)$. The entry $c$ is constant. To study the orbit of this field under an automorphism of $E_{-1}^{\,1}$, we take a general automorphism

$$\psi = \begin{pmatrix} 1 & d \\ 0 & e \end{pmatrix},$$

in which $d$ is a section of $\mathcal{O}(2)$ and $e \in \mathbb{C}^*$. Under $\psi$, the Higgs field is sent to

$$\phi' = \psi\phi\psi^{-1} = \begin{pmatrix} a+d & -2ade^{-1} + be^{-1} - d^2e^{-1} \\ e & -a-d \end{pmatrix}.$$

Taking the transformation $\psi$ with $e = 1$, $d = -a$, we get

$$\phi' = \psi\phi\psi^{-1} = \begin{pmatrix} 0 & -2ad + b - d^2 \\ 1 & 0 \end{pmatrix}.$$

In other words, the conjugacy class of a trace-free Higgs field acting on $E_{-1}^1$ is determined by a unique section $\rho = -2ad + b - d^2 = -\det\phi \in H^0(\mathbb{P}^1; \mathcal{O}(4))$. □

**Remark.** Riemann-Roch tells us for the direct image of a line bundle $L$ on $X_\rho$ to be a rank-2 vector bundle of degree 0 on $\mathbb{P}^1$, then we must have $\deg L = 2$. On $\mathbb{P}^1$, twisting $E_0$ by $\mathcal{O}(-1)$ gives $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, which has no sections. On the other hand, twisting $E_{-1}^1$ by $\mathcal{O}(-1)$ gives $\mathcal{O} \oplus \mathcal{O}(-2)$, which still has a section. Because the direct image functor preserves the number of global sections, this is the same as asking whether or not $L \otimes \pi_\rho^*\mathcal{O}(-1)$ has sections. The twisted line bundle $L \otimes \pi_\rho^*\mathcal{O}(-1)$ has degree $\deg L + (-1)\deg \pi_\rho = 2 - 2 = 0$. The only line bundle of degree 0 with a section is the trivial line bundle. Pushing down the trivial line bundle therefore produces the co-Higgs bundle $(E_{-1}^1, Q(\rho))$, while pushing down any other line bundle gives a Higgs field for $E_0$.

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Mathematical Institute, 24-29 St. Giles', Oxford, UK OX1 3LB.
*E-mail address*: steven.rayan@maths.ox.ac.uk